# SUBWORD COUNTING AND THE INCIDENCE ALGEBRA

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ABSTRACT. The Pascal matrix, $P$, is an upper diagonal matrix whose entries are the binomial coefficients. In 1993 Call and Velleman demonstrated that it satisfies the beautiful relation $P = \exp(H)$ in which $H$ has the numbers 1, 2, 3, etc. on its superdiagonal and zeros elsewhere. We generalize this identity to the incidence algebras $I(A^*)$ and $I(\mathcal{S})$ of functions on words and permutations, respectively. In $I(A^*)$ the entries of $P$ and $H$ count subwords; in $I(\mathcal{S})$ they count permutation patterns. Inspired by vincular permutation patterns we define what it means for a subword to be restricted by an auxiliary index set $R$; this definition subsumes both factors and (scattered) subwords. We derive a theorem for words corresponding to the Reciprocity Theorem for patterns in permutations: Up to sign, the coefficients in the Mahler expansion of a function counting subwords restricted by the set $R$ is given by a function counting subwords restricted by the complementary set $R^c$.

## 1. INTRODUCTION

Let us start by recalling some terminology from the theory of posets; our presentation follows that of Stanley [8, sec. 3]. Let $(Q, \leqslant)$ be a poset. For $x, y \in Q$, let $[x, y] = \{z : x \leqslant z \leqslant y\}$ denote the *interval* between $x$ and $y$. All posets considered in this paper will be *locally finite*: that is, each interval $[x, y]$ is finite. We say that $y$ *covers* $x$ if $[x, y] = \{x, y\}$.

Let $\mathrm{Int}(Q) = \{(x, y) \in Q \times Q : x \leqslant y\}$; it is a set that can be thought of as representing the intervals of $Q$. The *incidence algebra*, $I(Q)$, of $(Q, \leqslant)$ over some field $K$ is the $K$-algebra of all functions $F : \mathrm{Int}(Q) \to K$ with the usual structure as a vector space over $K$ and multiplication (convolution) defined by

$$(FG)(x, y) = \sum_{x \leqslant z \leqslant y} F(x, z)G(z, y),$$

and identity, $\delta$, defined by $\delta(x, y) = 1$ if $x = y$, and $\delta(x, y) = 0$ if $x \neq y$. Let $[\![\psi]\!]$ be the Iverson bracket; it denotes a number that is 1 if the statement $\psi$ is satisfied, and 0 otherwise. With this notation we have $\delta(x, y) = [\![x = y]\!]$. Two other prominent elements of the incidence algebra are $\zeta(x, y) = 1$ and $\eta(x, y) = [\![y \text{ covers } x]\!]$. One can show (see Stanley [8, sec. 3]) that $f \in I(Q)$ has a two-sided inverse $f^{-1}$ if and only if $f(x, x) \neq 0$ for all $x \in Q$. For instance, $\zeta$ is invertible and its inverse, $\mu = \zeta^{-1}$, is known as the *Möbius function*.

The $(i, j)$ entry of the upper triangular $n \times n$ *Pascal matrix* $P_n$ is defined as $\binom{j}{i}$. The superdiagonal matrix $H_n$ is defined by letting the entries on the superdiagonal $j = i + 1$ be $j$. Further, let $I_n$ denote the $n \times n$ identity matrix. In 1993 Call

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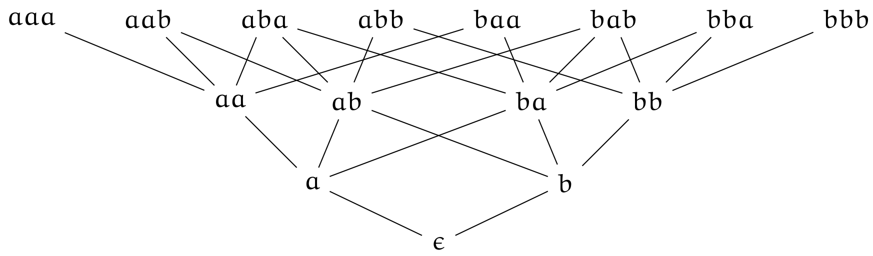


FIGURE 1. The first four levels of the poset of words over $\{a, b\}$ with respect to the subword/division order.

and Velleman [4] calculated the powers of $H_n$ and showed that $e^{H_n} = P_n$, where $e^{H_n} = I_n + H_n + H_n^2/2! + H_n^3/3! + \cdots$. As an example, for $n = 3$ we have

$$\exp\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 2 & 0 \\ & & 0 & 3 \\ & & & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 3 \\ & & 1 & 3 \\ & & & 1 \end{bmatrix}.$$

If we consider $\mathbb{N}$ as a poset under the usual order on integers we may view the result of Call and Velleman as an identity in the incidence algebra $I(\mathbb{N})$: We have

$$e^H = P, \tag{1}$$

where $H(i, j) = j\eta(i, j) = j[\![j = i + 1]\!]$ and $P(i, j) = \binom{j}{i}$ are elements of $I(\mathbb{N})$, and

$$e^H = \delta + H + \frac{1}{2!}H^2 + \frac{1}{3!}H^3 + \cdots.$$

Note that the restriction of $H$ to integers smaller than some fixed bound is nilpotent, and hence each entry in $e^H$ is a finite sum (in fact, a single term). We shall generalize (1) to the incidence algebra over a poset of words, but first we need a few more definitions.

Let $[n] = \{1, 2, \ldots, n\}$. Let $A^*$ be the set of words with letters in a given finite set (alphabet) $A$. A word $u = a_1 a_2 \ldots a_k$, with $a_i \in A$, is said to be a *subword* of a word $v = b_1 b_2 \ldots b_n$, with $b_i \in A$, if there is an order-preserving injection $\varphi : [k] \to [n]$ such that $a_i = b_{\varphi(i)}$ for all $i \in [k]$; that is, if $u$ is a subsequence of $v$. By an *occurrence* of $u$ (as a subword) in $v$ we shall mean the function $\varphi$, often represented by the set $\{\varphi(i) : i \in [k]\}$. For instance, the word $baacbab$ contains four occurrences of $aab$. Following Eilenberg [5, VIII.10] we let $\binom{v}{u}$ denote the number of occurrences of $u$ as a subword in $v$. Note that if $a$ is a letter, then $\binom{a^n}{a^k} = \binom{n}{k}$, where the right-hand side is the ordinary binomial coefficient.

We will consider $A^*$ a poset under the *subword order*, also called division order. That is, $u \leqslant v$ in $A^*$ if and only if $\binom{v}{u} > 0$. For $A = \{a, b\}$ the first four levels of the Hasse diagram of this infinite poset are depicted in Figure 1.

We are now ready to state the promised generalization of (1). Recall that $\eta(x, y) = [\![y \text{ covers } x]\!]$. For the poset $\mathbb{N}$ we hence have $\eta(i, j) = 1$ if, and only if, $j = i + 1$. For the poset $A^*$ we have $\eta(u, v) = 1$ if, and only if, $u$ is a subword of $v$ and $|v| = |u| + 1$. Theorem 2, which we prove in Section 2, then states the following:

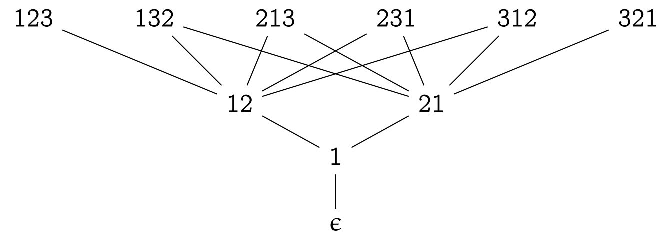


FIGURE 2. The first four levels of the poset of permutations with respect to pattern containment.

> Define the elements $P$ and $H$ of the incidence algebra $I(A^*)$ by $P(u,v) = \binom{v}{u}$ and $H(u,v) = \binom{v}{u}\eta(u,v)$. Then $P = e^H$.

Note that if $A$ is a singleton, and $u$ and $v$ are words in $A^*$ of length $n$ and $k$, respectively, then $u \leqslant v$ in $A^*$ precisely then $k \leqslant n$ in $\mathbb{N}$. Further, $P(u,v) = \binom{n}{k}$ and $H(u,v) = \binom{n}{k}[\![n = k+1]\!] = n[\![n = k+1]\!]$. Thus, Equation 1 is a special case of Theorem 2.

We can also use the relation $P = e^H$ to derive a simple formula for the powers of $P$. The following statement is Theorem 8 of Section 3.

> For $u, v \in A^*$ and any integer $d$ we have $P^d(u,v) = d^{|v|-|u|}P(u,v)$.

Let us for a moment turn to permutations instead of words. Let $\mathcal{S}_n$ be the set of permutations of $[n]$, and let $\mathcal{S} = \cup_{n\geqslant 0}\mathcal{S}_n$. Two sequences of integers $a_1 \dots a_k$ and $b_1 \dots b_k$ are *order-isomorphic* if for every $i, j \in [k]$ we have $a_i < a_j \Leftrightarrow b_i < b_j$. A permutation $\sigma \in \mathcal{S}_k$ is said to be contained in a permutation $\pi \in \mathcal{S}_n$ if there is an order-preserving injection $\varphi : [k] \to [n]$ such that $\pi(\varphi(1))\pi(\varphi(2))\dots\pi(\varphi(k))$ is order-isomorphic to $\sigma$. In this context, $\sigma$ is often called a *pattern*. By an *occurrence* of $\sigma$ in $\pi$ we shall mean the function $\varphi$, or the set $\{\varphi(i) : i \in [k]\}$. For instance, the permutation 43152 contains two occurrences of the pattern 231, namely $\{1,4,5\}$ and $\{2,4,5\}$. Let $\binom{\pi}{\sigma}$ be the number of occurrences of $\sigma$ in $\pi$. Note that $\binom{12\dots n}{12\dots k} = \binom{n}{k}$, where the right-hand side is the ordinary binomial coefficient.

Much like with $A^*$ we endow $\mathcal{S}$ with a poset structure by postulating that $\sigma \leqslant \pi$ if, and only if, $\binom{\pi}{\sigma} > 0$. The first four levels of the Hasse diagram of this infinite poset are depicted in Figure 2. It turns out that the $P = e^H$ equation in $I(A^*)$ has a natural counterpart in $I(\mathcal{S})$:

> Define the elements $P$ and $H$ of the incidence algebra $I(\mathcal{S})$ by $P(\sigma,\pi) = \binom{\pi}{\sigma}$ and $H(\sigma,\pi) = \binom{\pi}{\sigma}\eta(\sigma,\pi)$. Then $P = e^H$.

In 2011 Brändén and the present author (see [3]) presented a "Reciprocity Theorem" for so called *mesh patterns*, a class of permutation patterns introduced in the same paper. Intuitively, an occurrence of a mesh pattern is an occurrence in the sense defined above with additional restrictions on the relative position of the entries of the occurrence. *Vincular patterns* (Babson and Steingrímsson, 2000 [1]) as well as *bivincular patterns* (Bousquet-Mélou et al., 2010 [2]) can be seen as special mesh patterns. In this paper we shall not need the general definition of a mesh pattern, rather we just give the more specialized definition of a vincular pattern:

Let $\pi \in \mathcal{S}_n$. Let $\sigma \in \mathcal{S}_k$ and $R \subseteq [0,k]$. The pair $p = (\sigma, R)$ is called a *vincular pattern* and an occurrence of $p$ in $\pi$ is an order-preserving injection $\varphi : [k] \to [n]$ such that $\pi(\varphi(1))\pi(\varphi(2))\dots\pi(\varphi(k))$ is order-isomorphic to $\sigma$ and, for all $i \in R$, $\varphi(i+1) = \varphi(i)+1$, where $\varphi(0) = 0$ and $\varphi(k+1) = n+1$ by convention. Let $\binom{\pi}{p}$ denote the number of occurrences of $p$ in $\pi$. We will find it convenient to write $\binom{\pi}{\sigma,R}$ rather than the typographically awkward $\binom{\pi}{(\sigma,R)}$. For instance,

$$\binom{43152}{231,\{0\}} = 1,\ \binom{43152}{231,\{1\}} = 0,\ \binom{43152}{231,\{2\}} = 2 \text{ and } \binom{43152}{231,\{0,3\}} = 1.$$

Any function $f : \mathcal{S} \to \mathbb{Q}$ can be written as a unique, but typically infinite, linear combination of functions $\binom{\cdot}{\sigma}$. To see this, let $I(\mathcal{S})$ act on the function space $\mathbb{Q}^{\mathcal{S}}$ by

$$(f * F)(\pi) = \sum_{\sigma \leqslant \pi} f(\sigma)F(\sigma,\pi).$$

Thus, $f = \sum_\sigma c(\sigma)\binom{\cdot}{\sigma}$ is equivalent to $f = c*P$ and, since $P$ is invertible, $c = f*P^{-1}$. Viewing $\binom{\cdot}{p}$ as a function in $\mathbb{Q}^{\mathcal{S}}$, the Reciprocity Theorem [3, Thm. 1] then gives the coefficient $c(\sigma)$ of $\binom{\cdot}{\sigma}$ in terms of occurrences of the *dual* pattern $p^\star = (\sigma, [0,|\sigma|]\setminus R)$. For any permutation $\pi$, we have

$$\binom{\pi}{p} = \sum_{\upsilon \in A^*} \langle p, \upsilon\rangle \binom{\pi}{\upsilon}, \quad \text{where } \langle p,\upsilon\rangle = (-1)^{|\upsilon|-|\sigma|}\binom{\upsilon}{p^\star}.$$

In Section 4 we present a corresponding theorem for words. To make this explicit we need a definition: Let $u = a_1a_2\dots a_k$, with $a_i \in A$, and let $v = b_1b_2\dots b_n$, with $b_i \in A$. Let $p = (u, R)$ with $R \subseteq [0,k]$. An *occurrence* of $p$ in $v$ is an order-preserving injection $\varphi : [k] \to [n]$ such that $a_i = b_{\varphi(i)}$ for all $i \in [k]$, and, for all $i \in R$, $\varphi(i+1) = \varphi(i)+1$, with the convention that $\varphi(0) = 0$ and $\varphi(k+1) = n+1$. Let $\binom{v}{p}$ denote the number of occurrences of $p$ in $v$. For example,

$$\binom{baacbab}{aab,\{0\}} = 0,\ \binom{baacbab}{aab,\{1\}} = 2,\ \binom{baacbab}{aab,\{1,2\}} = 0 \text{ and } \binom{baacbab}{aab,\{1,3\}} = 1.$$

We would like to draw attention to the following three important special cases of the above definition:

1. if $R = \emptyset$ then $\binom{v}{p} = \binom{v}{u}$;
2. if $R = [1, k-1]$ then $\binom{v}{p}$ is the number of occurrences of $u$ as a factor in $v$;
3. if $R = [0,k]$ then $\binom{v}{p} = \delta(u,v)$.

With regard to the second item, a word $u$ is a *factor* of another word $v$ if there are (possibly empty) words $x$ and $y$ such that $v = xuy$.

We are now ready to state the Reciprocity Theorem for $I(A^*)$, which is Theorem 12 of Section 4. For $u$ a word and $R \subseteq [0,|u|]$ let $p = (u,R)$, and let $p^\star = (u, [0,|u|]\setminus R)$. Then, for any word $w$, we have

$$\binom{w}{p} = \sum_{v \in A^*} \langle p, v\rangle \binom{w}{v}, \quad \text{where } \langle p,v\rangle = (-1)^{|v|-|u|}\binom{v}{p^\star}.$$

## 2. P IS THE EXPONENTIAL OF H

Let $P$ and $H$, in $I(A^*)$, be defined as in the introduction. The key to proving that $P = e^H$ will be the simple formula for the powers of $H$ provided by the Lemma 1 below. Its proof will use some terminology we now define: A subset $C$ of a poset $Q$ is called a chain if for any pair of elements $x$ and $y$ in the subposet $C$ of $Q$ we have $x \leqslant y$ or $y \leqslant x$. A chain $x_0 < x_1 < \cdots < x_n$ is *saturated* if $x_i$ covers $x_{i-1}$ for each $i \in [n]$.

**Lemma 1.** *For* $u, v \in A^*$ *we have*

$$H^\ell(u, v) = \ell! \binom{v}{u} [\![\, |v| = |u| + \ell \,]\!]. \tag{2}$$

*Proof.* We shall give a combinatorial proof. Let $n = |v|$. Expanding the left-hand side, $H^\ell(u, v)$, we get

$$\sum H(x_0, x_1) H(x_1, x_2) \ldots H(x_{\ell-1}, x_\ell),$$

where the sum is over all saturated chains $u = x_0 < x_1 < \cdots < x_{\ell-1} < x_\ell = v$. Assume that we are given such a chain. By transitivity we may consider each $x_i$ as a subword of $v$; let $\mathfrak{o}_i \subseteq [n]$ be a set of indices that determine the subword $x_i$ in $v$. Because the chain $x_0 < x_1 < \cdots < x_\ell$ is saturated, so is the chain $\mathfrak{o}_0 \subset \mathfrak{o}_1 \subset \cdots \subset \mathfrak{o}_\ell$, and thus $\mathfrak{o}_{i+1} \setminus \mathfrak{o}_i$ is a singleton. Suppose that $\mathfrak{o}_{i+1} \setminus \mathfrak{o}_i = \{m_i\}$. Then $m_0 m_1 \ldots m_{\ell-1}$ is a permutation of $\mathfrak{o}_\ell \setminus \mathfrak{o}_0$. To summarize, any given saturated chain $u = x_0 < x_1 < \cdots < x_{\ell-1} < x_\ell = v$ together with occurrences $\mathfrak{o}_0, \mathfrak{o}_1, \ldots, \mathfrak{o}_\ell$ of $x_0, x_1, \ldots, x_\ell$, respectively, in $v$, determine a pair consisting of a permutation of the $\ell$ elements of $\mathfrak{o}_\ell \setminus \mathfrak{o}_0$ and an occurrence $\mathfrak{o}_0$ of $u$ in $v$.

Conversely, let $\mathfrak{o}_\ell = [n]$ be the index set of $v$, and let $\mathfrak{o}_0$ be an occurrence of $u$ in $v$. Let $m_0 m_1 \ldots m_{\ell-1}$ be a permutation of $\mathfrak{o}_\ell \setminus \mathfrak{o}_0$. We construct a saturated chain $u = x_0 < x_1 < \cdots < x_{\ell-1} < x_\ell = v$ and occurrences $\mathfrak{o}_0, \mathfrak{o}_1, \ldots, \mathfrak{o}_\ell$ of $x_0, x_1, \ldots, x_\ell$, respectively, in $v$ by letting $\mathfrak{o}_{i+1} = \mathfrak{o}_i \cup \{m_i\}$. □

**Theorem 2.** *Define the elements* $P$ *and* $H$ *of the incidence algebra* $I(A^*)$ *by* $P(u, v) = \binom{v}{u}$ *and* $H(u, v) = \binom{v}{u}\eta(u, v)$*. Then* $P = e^H$*.*

*Proof.* By Lemma 1 we have

$$(e^H)(u, v) = \sum_{\ell \geqslant 0} \frac{1}{\ell!} H^\ell(u, v) = \sum_{\ell \geqslant 0} \binom{v}{u} [\![\, |v| = |u| + \ell \,]\!] = \binom{v}{u}. \qquad \square$$

**Example 3.** Let us illustrate Lemma 1 and its proof for $A = \{a, b\}$, $\ell = 2$, $u = ab$ and $v = aaba$. The left-hand side of (2) is

$$H(ab, aab) H(aab, aaba) + H(ab, aba) H(aba, aaba) = 2 \cdot 1 + 1 \cdot 2,$$

while the right-hand side is $2! \binom{aaba}{ab} = 2 \cdot 2$. The following table gives the saturated chains and corresponding pairs as in the proof of Lemma 1:

$$\begin{array}{ll}
\{1,3\} \subset \{1,2,3\} \subset \{1,2,3,4\} & (24, \{1,3\}); \\
\{2,3\} \subset \{1,2,3\} \subset \{1,2,3,4\} & (14, \{2,3\}); \\
\{1,3\} \subset \{1,3,4\} \subset \{1,2,3,4\} & (42, \{1,3\}); \\
\{2,3\} \subset \{2,3,4\} \subset \{1,2,3,4\} & (41, \{2,3\}).
\end{array}$$

**Example 4.** If we only consider words shorter than some fixed length, then the poset of words is finite and the elements of the incidence algebra can be seen as upper triangular matrices. In the following example we consider words over $\{a, b\}$ of length at most 2. Then

$$H = \begin{bmatrix} 0 & \binom{a}{\epsilon} & \binom{b}{\epsilon} & 0 & 0 & 0 & 0 \\ & 0 & 0 & \binom{aa}{a} & \binom{ab}{a} & \binom{ba}{a} & 0 \\ & & 0 & 0 & \binom{ab}{b} & \binom{ba}{b} & \binom{bb}{b} \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 2 & 1 & 1 & 0 \\ & & 0 & 0 & 1 & 1 & 2 \\ & & & 0 & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \\ & & & & & & 0 \end{bmatrix}.$$

Note that all eigenvalues of $H$ are zero and thus $H$ is nilpotent. To be more precise, $H^i$ is the zero matrix for all $i \geqslant 3$, and

$$e^H = I + H + \frac{1}{2}H^2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 1 & 0 & 2 & 1 & 1 & 0 \\ & & 1 & 0 & 1 & 1 & 2 \\ & & & 1 & 0 & 0 & 0 \\ & & & & 1 & 0 & 0 \\ & & & & & 1 & 0 \\ & & & & & & 1 \end{bmatrix},$$

which, in agreement with Theorem 2, is equal to

$$\begin{bmatrix} \binom{\epsilon}{\epsilon} & \binom{a}{\epsilon} & \binom{b}{\epsilon} & \binom{aa}{\epsilon} & \binom{ab}{\epsilon} & \binom{ba}{\epsilon} & \binom{bb}{\epsilon} \\ & \binom{a}{a} & 0 & \binom{aa}{a} & \binom{ab}{a} & \binom{ba}{a} & 0 \\ & & \binom{b}{b} & 0 & \binom{ab}{b} & \binom{ba}{b} & \binom{bb}{b} \\ & & & \binom{aa}{aa} & 0 & 0 & 0 \\ & & & & \binom{ab}{ab} & 0 & 0 \\ & & & & & \binom{ba}{ba} & 0 \\ & & & & & & \binom{bb}{bb} \end{bmatrix}.$$

The powerset of $[n]$ with respect to the subset relation is a poset called the *boolean algebra* and is denoted $B_n$. We note that one part of the proof of Lemma 1 above is the well known fact that the number of saturated chains from $\emptyset$ to $[n]$ in $B_n$ is $n!$. Recall that $\zeta(S, T) = 1$ and $\eta(S, T) = [\![T \text{ covers } S]\!]$. The proof of the following proposition, which we omit, is very similar to but slightly easier than the proof of Lemma 1.

**Proposition 5.** *In the incidence algebra* $I(B_n)$ *we have* $e^\eta = \zeta$.

The simple formula $\mu(S, T) = (-1)^{|T \setminus S|}$ for the Möbius function of the boolean algebra can be derived from the isomorphism $B_n \cong \mathbf{2}^n$ using the so called product rule [8, Ex. 3.8.3]. This formula also follows from Proposition 5: Let $S, T \in B_n$. As in the proof of Lemma 1, the number of saturated chains starting at $S$ and ending in $T$ is $\ell!$, where $\ell = |T \setminus S|$. The number of such chains is also, however, $\eta^\ell(S, T)$.

To be more precise $\eta^\ell(S,T) = \ell! [\![ |T| = |S| + \ell ]\!]$, and thus

$$\mu(S,T) = \zeta^{-1}(S,T) = (e^{-\eta})(S,T) = \sum_{\ell \geqslant 0} \frac{(-1)^\ell}{\ell!} \eta^\ell(S,T) = (-1)^\ell.$$

Let us now consider the permutation poset $\mathcal{S}$. It is easy to see that the proof of Lemma 1 can be modified to apply to the setting of the incidence algebra $I(\mathcal{S})$, and we close this section by stating the counterpart of Theorem 2 that follows from doing so.

**Theorem 6.** *Define the elements* $P$ *and* $H$ *of the incidence algebra* $I(\mathcal{S})$ *by* $P(\sigma,\pi) = \binom{\pi}{\sigma}$ *and* $H(\sigma,\pi) = \binom{\pi}{\sigma}\eta(\sigma,\pi)$*. Then* $P = e^H$*.*

**Example 7.** If we only consider permutations shorter than some fixed length, then the poset of words is finite and the elements of the incidence algebra can be seen as upper triangular matrices. In the following example we consider permutations of length at most 2. We have

$$H = \begin{bmatrix} 0 & \binom{1}{\epsilon} & 0 & 0 \\ & 0 & \binom{12}{1} & \binom{21}{1} \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & 2 & 2 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} \quad \text{and}$$

$$P = I + H + \frac{1}{2}\begin{bmatrix} 0 & 0 & 2 & 2 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ & 1 & 2 & 2 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} = \begin{bmatrix} \binom{\epsilon}{\epsilon} & \binom{1}{\epsilon} & \binom{12}{\epsilon} & \binom{21}{\epsilon} \\ & \binom{1}{1} & \binom{12}{1} & \binom{21}{1} \\ & & \binom{12}{12} & 0 \\ & & & \binom{21}{21} \end{bmatrix}.$$

## 3. Powers of P

From Sakarovitch and Simon [6, Corollary 6.3.8] we learn that

$$\sum_{w \in A^*} (-1)^{|u|+|v|} \binom{w}{u} \binom{v}{w} = \delta(u,v). \tag{3}$$

Their proof uses the so called Magnus transformation, the algebra endomorphism $a \mapsto a+1$ of $\mathbb{Z}\langle A \rangle$. An analog of this result for permutations is a consequence of the Reciprocity Theorem for mesh patterns [3, Corollary 2]; more recently, Vargas [9] has given a proof of the same result using an analog of the Magnus transformation for permutations.

In the context of the incidence algebra, Equation 3 states that $P^{-1}(u,v) = (-1)^{|v|-|u|}P(u,v)$. We could use Theorem 2 to give a simple alternative proof of this. Indeed, $P = e^H$ implies that $P^{-1} = e^{-H}$. We can in fact prove something stronger:

**Theorem 8.** *For any words* $u$ *and* $v$ *in* $A^*$*, and any integer* $d$*, we have*

$$P^d(u,v) = d^{|v|-|u|}P(u,v).$$

*If* $d \neq 0$ *and* $D_d(u,v) = d^{|u|}\delta(u,v)$*, then an equivalent way of stating this result is* $P^d = D_d^{-1} P D_d = D_{1/d} P D_d$*. In particular,* $P^{-1} = D_{-1} P D_{-1}$*.*

*Proof.* We have

$$
\begin{aligned}
P^d(u,v) &= (e^H)^d(u,v) && \text{by Theorem 2}\\
&= e^{dH}(u,v)\\
&= \sum_{\ell\geqslant 0}\frac{1}{\ell!}(dH)^\ell(u,v)\\
&= \sum_{\ell\geqslant 0}\frac{d^\ell}{\ell!}H^\ell(u,v)\\
&= \sum_{\ell\geqslant 0} d^\ell P(u,v)[\![\,|v| = |u| + \ell\,]\!] && \text{by Lemma 1}\\
&= d^{|v|-|u|}P(u,v). && \square
\end{aligned}
$$

**Example 9.** If we restrict $P$ to words of length at most $n$ and let $A$ be a singleton alphabet, then the function $P$ reduces to the $n \times n$ Pascal matrix $P_n$. Thus, a corollary to Theorem 8 is that the Pascal matrix satisfies $P_n^d(i,j) = d^{j-i}P_n(i,j)$. This is, however, a known result due to Call and Velleman [4].

A *multichain* in a poset $Q$ is a multiset whose underlying set is a chain in $Q$. It is well known, and easy to prove, that the number of multichains

$$\emptyset = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{d-1} \subseteq S_d = [\ell]$$

in the boolean algebra $B_\ell$ is $d^\ell$. Indeed, such a multichain is uniquely specified by a function $t : [\ell] \to [d]$ where $t(i)$ is the smallest $j \in [d]$ for which $i \in S_j$. We use this fact in the following alternative proof of Theorem 8.

*Combinatorial proof of Theorem 8.* Let $n = |v|$ and $\ell = |v| - |u|$. Expanding the left-hand side, $P^d(u,v)$, we get

$$\sum P(x_0,x_1)P(x_1,x_2)\ldots P(x_{d-1},x_d),$$

where the sum is over all multichains $u = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{d-1} \leqslant x_d = v$. Assume that we are given such a chain. By transitivity we may consider each $x_i$ as a subword of $v$; let $\mathfrak{o}_i \subseteq [n]$ be a set of indices that determine the subword $x_i$ in $v$. Let $S_0 = \mathfrak{o}_0 \setminus \mathfrak{o}_0 = \emptyset, S_1 = \mathfrak{o}_1 \setminus \mathfrak{o}_0$, $S_2 = \mathfrak{o}_2 \setminus \mathfrak{o}_0$, etc, and let $S = \mathfrak{o}_d \setminus \mathfrak{o}_0$. Then $\emptyset = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{d-1} \subseteq S_d = S$ is a multichain in the boolean algebra on $S$. As in the paragraph preceding this proof, let $t : S \to [d]$ be the function specifying that multichain. Because $|S| = \ell$, the number of such functions, and therefore also the number of multichains, is $d^\ell$.

To summarize, any given multichain $u = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_{d-1} \leqslant x_d = v$ together with occurrences $\mathfrak{o}_0, \mathfrak{o}_1, \ldots, \mathfrak{o}_d$ of $x_0, x_1, \ldots, x_d$, respectively, in $v$, determine a pair consisting of a function $t : \mathfrak{o}_d \setminus \mathfrak{o}_0 \to [d]$ and an occurrence $\mathfrak{o}_0$ of $u$ in $v$. The total number of such pairs is, of course, $d^{|v|-|u|}P(u,v)$. It is also easy to see how this could be reversed and thus the procedure described is a bijection. $\square$

Either of the two proofs we have presented for Theorem 8 can easily be adopted to the permutation setting. Thus we have rediscovered the following result, which was known to Petter Brändén already in 2002 (personal communication).

**Theorem 10** (Brändén 2002)**.** *For $\sigma, \pi \in \mathcal{S}$ and any integer $d$ we have $P^d(\sigma,\pi) = d^{|\pi|-|\sigma|}P(\sigma,\pi)$.*

## 4. A reciprocity theorem

Let $u = a_1 a_2 \dots a_k$, with $a_i \in A$, and let $v = b_1 b_2 \dots b_n$, with $b_i \in A$. Let $p = (u, R)$ with $R \subseteq [0, k]$. Recall that an *occurrence* of $p$ in $v$ is an order-preserving injection $\varphi : [k] \to [n]$ such that $a_i = b_{\varphi(i)}$ for all $i \in [k]$, and, for all $i \in R$, $\varphi(i+1) = \varphi(i)+1$, with the convention that $\varphi(0) = 0$ and $\varphi(k+1) = n+1$. The number of occurrences of $p$ in $v$ is denoted by $\binom{v}{p}$ or $\binom{v}{u,R}$.

**Lemma 11** (A generalization of Pascal's formula)**.** *Let $u$ and $v$ be words, and let $a$ and $b$ be letters. Let $k = |ub|$ and let $R \subseteq [0, k]$. Then we have*

$$\binom{va}{ub, R} = [\![k \notin R]\!]\binom{v}{ub, R} + [\![a = b]\!]\binom{v}{u, R \setminus \{k\}}.$$

*Proof.* An occurrence of $(ub, R)$ in $va$ may match the last letter of $ub$ with the last letter of $va$; the number of such occurrences is $[\![a = b]\!]\binom{v}{u,R\setminus\{k\}}$. If $k \in R$ this is the only option: we have to match the last letter of $ub$ with the last letter of $va$. If, on the other hand, $k \notin R$, then we have additional occurrences, namely those that do not involve the last letter of $va$, and there are $\binom{v}{ub,R}$ such occurrences. □

**Theorem 12** (Reciprocity)**.** *For $u$ a word and $R \subseteq [0, |u|]$ let $p = (u, R)$, and let $p^\star = (u, [0, |u|] \setminus R)$. Then, for any word $w$, we have*

$$\binom{w}{p} = \sum_{v \in A^*} \langle p, v\rangle \binom{w}{v}, \quad \textit{where } \langle p, v\rangle = (-1)^{|v|-|u|}\binom{v}{p^\star}.$$

*Proof.* We shall find it convenient to swap the role of $p$ and $p^\star$. That is, we shall prove the statement

$$\sum_{v \in A^*} (-1)^{|v|-|u|}\binom{v}{p}\binom{w}{v} = \binom{w}{p^\star}. \tag{4}$$

The proof will proceed by induction on the length of $w$. If $w = \epsilon$, the empty word, and $p = (u, R)$, then both sides of (4) are equal to $[\![u = \epsilon]\!]$. Assume that $w$ is nonempty and write $w = za$ with $z \in A^*$ and $a \in A$. If $p = (\epsilon, R)$ we have two cases to consider, namely $R = \emptyset$ and $R = \{0\}$. Clearly, $(\epsilon, \emptyset)^\star = (\epsilon, \{0\})$ and vice versa. Further, for $a \in A$, we have $\binom{za}{\epsilon,\{0\}} = 0$, $\binom{za}{\epsilon,\emptyset} = 1$, and

$$\sum_{v \in A^*} (-1)^{|v|}\binom{v}{\epsilon, R}\binom{za}{v}$$

evaluates to $\binom{za}{\epsilon} = 1$ if $R = \{0\}$; otherwise, that is, if $R = \emptyset$, it evaluates to

$$\sum_{v \in A^*} (-1)^{|v|}\binom{za}{v} = \sum_{k \geqslant 0}(-1)^k\binom{|za|}{k} = 0.$$

Let $b \in A$, $u \in A^*$, $k = |u| + 1$, and $p = (ub, R)$. In the following calculation we will use Lemma 11 and for convenience of notation we will refrain from subtracting $\{k\}$ from $R$ when writing the last term of the recursion in that lemma. That is, by convention $(u, R) = (u, R \cap [0, |u|])$ so that we disregard any part of $R$ that is outside

the interval $[0,|u|]$. Using induction, we have

$$\begin{aligned}
&\sum_{v\in A^*}(-1)^{|v|-k}\binom{v}{p}\binom{za}{v}\\
&\quad=\sum_{c\in A}\sum_{v\in A^*}(-1)^{1+|v|-k}\binom{vc}{ub,R}\binom{za}{vc}\\
&\quad=\sum_{c\in A}\sum_{v\in A^*}(-1)^{1+|v|-k}\binom{vc}{ub,R}\left(\binom{z}{vc}+[\![a=c]\!]\binom{z}{v}\right)\\
&\quad=\binom{z}{ub,R^c}+\sum_{v\in A^*}(-1)^{1+|v|-k}\binom{va}{ub,R}\binom{z}{v}\\
&\quad=\binom{z}{ub,R^c}+\sum_{v\in A^*}(-1)^{1+|v|-k}\left([\![k\notin R]\!]\binom{v}{ub,R}+[\![a=b]\!]\binom{v}{u,R}\right)\binom{z}{v}\\
&\quad=\binom{z}{ub,R^c}-[\![k\notin R]\!]\binom{z}{ub,R^c}+[\![a=b]\!]\binom{z}{u,R^c}\\
&\quad=[\![k\notin R^c]\!]\binom{z}{ub,R^c}+[\![a=b]\!]\binom{z}{u,R^c}\\
&\quad=\binom{za}{p^\star},
\end{aligned}$$

which completes the proof. □

**Example 13.** If $p=(u,[0,|u|])$ then $p^\star=(u,\emptyset)$ and $\binom{w}{p}=\delta(w,u)$. So, by the Reciprocity Theorem, $\delta(w,u)=\sum_{v\in A^*}(-1)^{|v|-|u|}\binom{v}{u}\binom{w}{v}$, giving us yet another proof of (3).

**Example 14.** In the introduction we remarked that any function $f:\mathcal{S}\to\mathbb{Q}$ can be expressed as a unique linear combination of functions $\binom{\cdot}{\sigma}$. Similarly, any function $f:A^*\to\mathbb{Q}$ can be written as a unique, but typically infinite, linear combination of functions $\binom{\cdot}{u}$: $I(A^*)$ acts on the right of the function space $\mathbb{Q}^{A^*}$ by $(f*F)(v)=\sum_{u\leqslant v}f(u)F(u,v)$, and thus $f=\sum_u c(u)\binom{\cdot}{u}$ is equivalent to $f=c*P$ and, since $P$ is invertible, $c=f*P^{-1}$. This is called the *Mahler expansion* of $f$ by Pin and Silva [7]. As an example, let $A=\mathbf{2}$, and define the parity function $\textsc{xor}:A^*\to\mathbf{2}$ by $\textsc{xor}(w)=[\![\,w$ has an odd number of ones$\,]\!]$. Then

$$\textsc{xor}(w)=\sum_{k\geqslant 1}(-2)^{k-1}\binom{w}{1^k}.$$

Indeed, assuming that $w\in A^*$ and $\ell=\binom{w}{1}$ we have

$$\textsc{xor}(w)=\sum_{k=1}^{\ell}(-2)^{k-1}\binom{\ell}{k}=\frac{1}{2}\big(1-(-1)^\ell\big)=\begin{cases}1 & \text{if } \ell \text{ is odd,}\\ 0 & \text{if } \ell \text{ is even.}\end{cases}$$

Similarly, it is easy to prove that if we define the two functions $\textsc{and},\textsc{or}:A^*\to\mathbf{2}$ by $\textsc{and}(w)=[\![\binom{w}{0}=0]\!]$ and $\textsc{or}(w)=[\![\binom{w}{1}>0]\!]$, then

$$\textsc{and}(w)=\sum_{k\geqslant 0}(-1)^k\binom{w}{0^k}\quad\text{and}\quad\textsc{or}(w)=\sum_{k\geqslant 1}(-1)^{k-1}\binom{w}{1^k}.$$

**Example 15.** Let $A=\{a,b,c\}$ and consider $p=(ac,\{1\})$. Note that $\binom{w}{p}$ is the number of occurrences of $ac$ as a *factor* in $w$. We shall now use the Reciprocity

Theorem to find the Mahler expansion of $\binom{\cdot}{p}$. We have $p^\star = (ac, \{0,2\})$ and $\binom{w}{p^*} = [\![\, w = avc \text{ for some } v \in A^* \,]\!]$. Thus,

$$\binom{w}{ac, \{1\}} = \sum_{v \in A^*} (-1)^{|v|} \binom{w}{avc}.$$

**Corollary 16.** *Let* $n$ *and* $k$ *be nonnegative integers. Let* $R \subseteq [0,k]$ *and* $R^c = [0,k] \setminus R$*. Then the Mahler expansion of the generalized binomial coefficient* $\binom{n}{k,R} = \#\big\{\{s_1, s_2, \ldots, s_k\} \subseteq [n] : s_{i+1} = s_i + 1 \text{ for } i \in R\big\}$*, with* $s_0 = 0$ *and* $s_{k+1} = n+1$*, in* $I(\mathbb{N})$ *is*

$$\binom{n}{k, R} = \sum_{\ell \geqslant 0} (-1)^{\ell - k} \binom{\ell}{k, R^c} \binom{n}{\ell}.$$

*Proof.* Let $A = \{a\}$, and let $u, v \in A^*$. Then $u = a^k$ and $v = a^n$ for some $k, n \in \mathbb{N}$. Let $R \subseteq [0,k]$. Then $\binom{v}{u,R} = \binom{n}{k,R}$, and the result follows from Theorem 12. □

**Example 17.** Let $n = 4$, $k = 2$ and $R = \{1\}$. By Corollary 16 we have

$$\binom{4}{2, \{1\}} = \binom{2}{2, \{0,2\}} \binom{4}{2} - \binom{3}{2, \{0,2\}} \binom{4}{3} + \binom{4}{2, \{0,2\}} \binom{4}{4}.$$

Here, the left-hand side is $\#\{\{1,2\},\{2,3\},\{3,4\}\} = 3$, and the right-hand side is $\#\{\{1,2\}\} \cdot 6 - \#\{\{1,3\}\} \cdot 4 + \#\{\{1,4\}\} \cdot 1 = 6 - 4 + 1 = 3$.

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